1.1 Using and understanding algorithms

- An **algorithm** is a **finite** sequence of step-by-step instructions carried out to solve a problem.

Algorithms can be given in words or in **flow charts**.

You need to be able to understand and use an algorithm given in words.

You have been using algorithms since you started school. Some examples of mathematical algorithms that you will be familiar with are:

- how to add several two-digit numbers
- how to multiply two two-digit numbers
- how to add, subtract, multiply or divide fractions.

It can be quite challenging to write a sequence of instructions for someone else to follow accurately.

Here are some more examples:

At the end of the street turn right and go straight over the crossroads, take the third left after the school, then ...

Affix base (B) to leg (A) using screw (F) and ...

Dice two large onions.
Slice 100 g mushrooms.
Grate 100 g cheese.

Example 1 SKILLS ANALYSIS

A 'happy' number is defined by the algorithm:

- write down any **integer**
- square its digits and find the sum of the squares
- repeat until either the answer is 1 (in which case the number is 'happy') or until you get trapped in a **cycle** (in which case the number is 'unhappy')

Show that:

a 70 is happy **b** 4 is unhappy

a $7^2 + 0^2 = 49$
$4^2 + 9^2 = 97$
$9^2 + 7^2 = 130$
$1^2 + 3^2 + 0^2 = 10$
$1^2 + 0^2 = 1$
so 70 is happy

b $4^2 = 16$
$1^2 + 6^2 = 37$
$3^2 + 7^2 = 58$
$5^2 + 8^2 = 89$
$8^2 + 9^2 = 145$
$1^2 + 4^2 + 5^2 = 42$
$4^2 + 2^2 = 20$
$2^2 + 0^2 = 4$
$4^2 = 16$
so 4 is unhappy

As soon as the sum of the squares matches a previous result, all of the steps in-between will be repeated, creating a cycle.

Watch out You will need to be able to understand, describe and apply specific algorithms in your exam. You do not need to learn any of the algorithms in this section.

Example 2 SKILLS INTERPRETATION

a Apply this algorithm.

1 Let $n = 1$, $A = 1$, $B = 1$.
2 Print A and B.
3 Let $C = A + B$.
4 Print C.
5 Let $n = n + 1$, $A = B$, $B = C$.
6 If $n < 5$, go to step **3**.
7 If $n = 5$, stop.

These are not equations.
They are instructions that mean:
- replace n by $n + 1$ (add 1 to n)
- A takes B's current value
- B takes C's current value

b Describe the numbers that are generated by this algorithm.

a Use a trace table.

A **trace table** is used to record the values of each variable as the algorithm is run.

Step	n	A	B	C	Print
1	1	1	1		
2					1, 1
3				2	
4					2
5	2	1	2		
6	Go to step 3				
3				3	
4					3
5	3	2	3		
6	Go to step 3				
3				5	
4					5
5	4	3	5		
6	Go to step 3				
3				8	
4					8
5	5	5	8		
6	Continue to step 7				
7	Stop				

You may be asked to complete a printed trace table in your exam. Just obey each instruction, in order.

b This algorithm produces the first few numbers in the Fibonnacci sequence.

You may be asked what the algorithm does.

Example 3 SKILLS INTERPRETATION

This algorithm multiplies the two numbers A and B.

1 Make a table with two columns.
Write A in the top row of the left-hand column and B in the top row of the right-hand column.
In the next row, write the values for A and B.

2 In successive rows, write:
- in the left-hand column, the number that is half of A, ignoring remainders
- in the right-hand column, the number that is double B

3 Repeat step **2** until you reach the row which has a 1 in the left-hand column.

4 Delete all rows where the number in the left-hand column is even.

5 Find the sum of the non-deleted numbers in the right-hand column.
This is the product AB.

This famous algorithm is sometimes called 'the Russian peasant's algorithm' or 'the Egyptian multiplication algorithm'.

Apply this algorithm when:

a $A = 29$ and $B = 34$ **b** $A = 66$ and $B = 56$

a

A	B
29	34
~~14~~	~~68~~
7	136
3	272
1	544
Total	986

So 29 × 34 = 986

In each row, the number in the left-hand column is halved and the number in the right-hand column is doubled.

Step **4** means that rows where the number in the left-hand column is even must be deleted before summing the right-hand column.

b

A	B
~~66~~	~~56~~
33	112
~~16~~	~~224~~
~~8~~	~~448~~
~~4~~	~~896~~
~~2~~	~~1792~~
1	3584
Total	3696

So 66 × 56 = 3696

Each deleted row has an even number in its left-hand column.

Exercise 1A SKILLS INTERPRETATION

1 Use the algorithm in Example 3 to evaluate:

a 244×125 **b** 125×244 **c** 256×123

2 The box below describes an algorithm.

> **1** Write the input numbers in the form $\frac{a}{b}$ and $\frac{c}{d}$.
> **2** Let $e = ad$.
> **3** Let $f = bc$.
> **4** Print $\frac{e}{f}$.

a Apply this algorithm with the input numbers $2\frac{1}{4}$ and $1\frac{1}{3}$.
b What does this algorithm do?

3 The box below describes an algorithm.

> **1** Let $A = 1$, $n = 1$.
> **2** Print A.
> **3** Let $A = A + 2n + 1$.
> **4** Let $n = n + 1$.
> **5** If $n \leqslant 10$, go to **2**.
> **6** Stop.

a Apply the algorithm.
b Describe the numbers produced by the algorithm.

P **4** The box below describes an algorithm.

> **1** Input A, r.
> **2** Let $C = \frac{A}{r}$ to 3 d.p.
> **3** If $|r - C| \leqslant 10^{-2}$ go to **7**.
> **4** Let $s = \frac{1}{2}(r + C)$ to 3 d.p.
> **5** Let $r = s$.
> **6** Go to **2**.
> **7** Print r.
> **8** Stop.

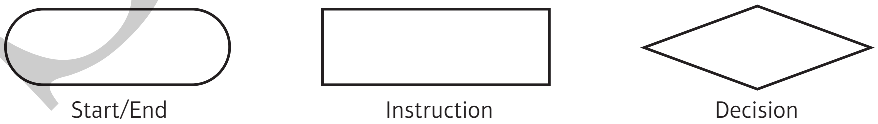
Hint This algorithm requires you to use the modulus function. If $x \neq y$, $|x - y|$ is the positive difference between x and y.
For example: $|3.2 - 7| = 3.8$.

a Use a trace table to apply the algorithm above when:
i $A = 253$ and $r = 12$ **ii** $A = 79$ and $r = 10$ **iii** $A = 4275$ and $r = 50$
b What does the algorithm produce?

1.2 Flow charts

You need to be able to apply an algorithm given as a flow chart.

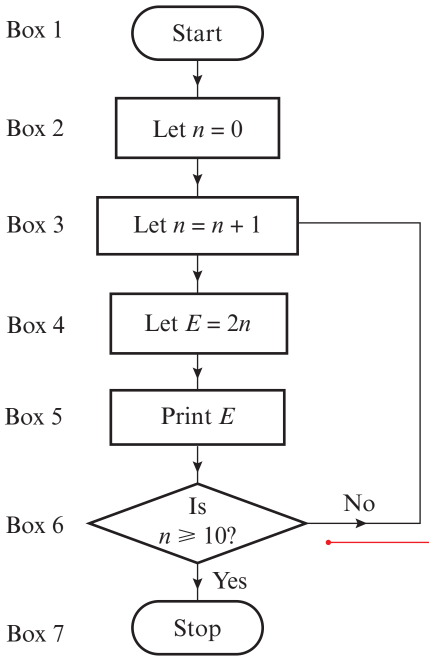
- In a flow chart, the shape of each box tells you about its function.



The boxes in a flow chart are linked by arrowed lines. As with an algorithm written in words, you need to follow each step in order.

Example 4 SKILLS ANALYSIS

The flow chart below describes an algorithm.



n is acting as a counter. It ensures that the algorithm stops after this **loop** has been completed 10 times.

A decision box will contain a question to which the answer is either 'yes' or 'no'.

a Apply this algorithm using a trace table.

b Alter box 4 to read 'Let $E = 3n$' and apply the algorithm again. How does this alter the output of the algorithm?

a

n	E	Box 6
0		
1	2	No
2	4	No
3	6	No
4	8	No
5	10	No
6	12	No
7	14	No
8	16	No
9	18	No
10	20	Yes

Output is 2, 4, 6, 8, 10, 12, 14, 16, 18, 20

b

n	E	Box 6
0		
1	3	No
2	6	No
3	9	No
4	12	No
5	15	No
6	18	No
7	21	No
8	24	No
9	27	No
10	30	Yes

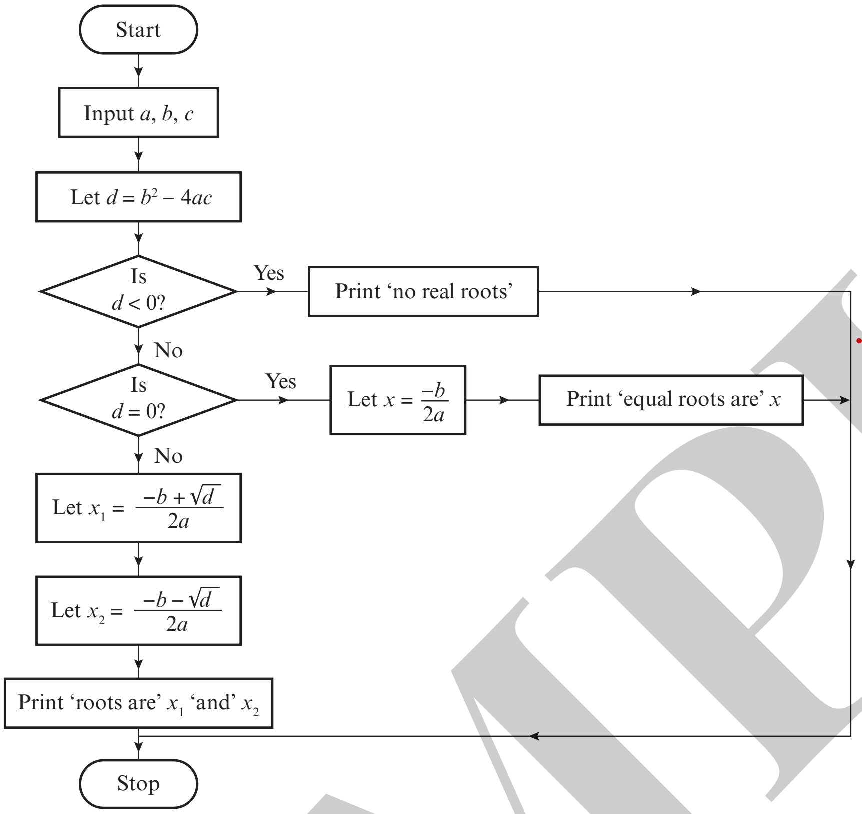
Output is 3, 6, 9, 12, 15, 18, 21, 24, 27, 30

This gives the first ten multiples of 3 rather than the first ten multiples of 2.

In a trace table each step must be made clear.

Example 5 SKILLS INTERPRETATION

This flow chart can be used to find the roots of an equation of the form $ax^2 + bx + c = 0$.



You should recognise d as the discriminant of the equation.

← Pure 1 Section 2.5

Demonstrate this algorithm for these equations:

a $6x^2 - 5x - 11 = 0$ **b** $x^2 - 6x + 9 = 0$ **c** $4x^2 + 3x + 8 = 0$

a

a	b	c	d	$d < 0$?	$d = 0$?	x_1	x_2
6	−5	−11	289	No	No	$\frac{11}{6}$	−1

roots are $\frac{11}{6}$ and −1

b

a	b	c	d	$d < 0$?	$d = 0$?	x
1	−6	9	0	No	Yes	3

equal roots are 3

c

a	b	c	d	$d < 0$?
4	3	8	−119	Yes

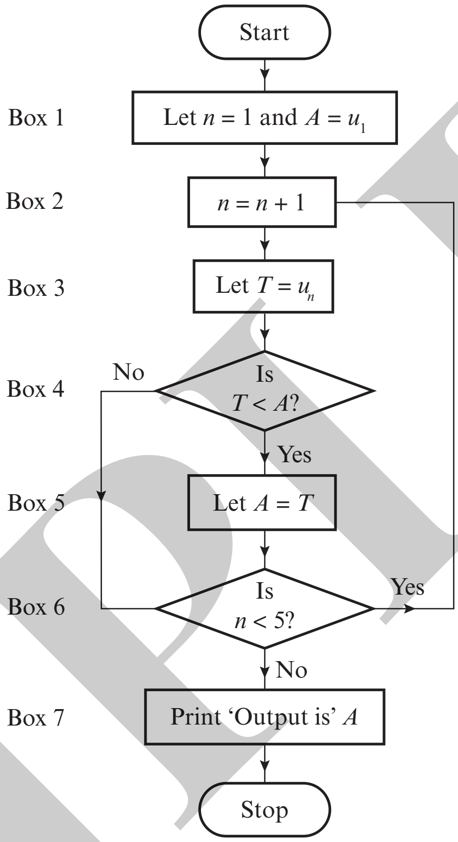
no real roots

Example 6 SKILLS INTERPRETATION

Apply the algorithm shown by the flow chart on the right to the data:

$u_1 = 10, u_2 = 15, u_3 = 9, u_4 = 7, u_5 = 11.$

What does the algorithm do?



	n	A	T	$T < A$?	$n < 5$?
Box 1	1	10			
Box 2	2				
Box 3			15		
Box 4				No	
Box 6					Yes
Box 2	3				
Box 3			9		
Box 4				Yes	
Box 5		9			
Box 6					Yes
Box 2	4				
Box 3			7		
Box 4				Yes	
Box 5		7			
Box 6					Yes
Box 2	5				
Box 3			11		
Box 4				No	
Box 6					No
Box 7	Output is 7				

The algorithm selects the smallest number from a list.

This is quite complicated because it has questions and a list of data. Tackle one step at a time.

The box numbers have been included to help you to follow the algorithm. You do not need to include them in your exam.

Exercise 1B SKILLS PROBLEM-SOLVING

1 Apply the flow chart in Example 5 to the following equations.

  a $4x^2 - 12x + 9 = 0$ **b** $-6x^2 + 13x + 5 = 0$ **c** $3x^2 - 8x + 11 = 0$

2 **a** Apply the flow chart in Example 6 to the following sets of data.

  i $u_1 = 28, u_2 = 26, u_3 = 23, u_4 = 25, u_5 = 21$

  ii $u_1 = 11, u_2 = 8, u_3 = 9, u_4 = 8, u_5 = 5$

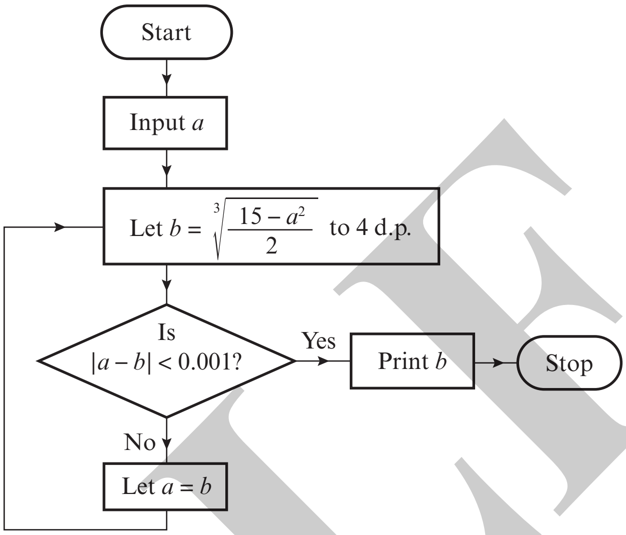
  b If box 4 is altered to Is $T > A$?, how will this affect the output?

  c Which box would need to be altered if the algorithm was to be applied to a list of 8 numbers?

3 The flow chart describes an algorithm that can be used to find the roots of the equation $2x^3 + x^2 - 15 = 0$.

a Use $a = 2$ to find a root of the equation.

b Use $a = 20$ to find a root of the equation. Comment on your answer.



E/P **4** The flow chart on the right describes how to apply Euclid's algorithm to two non-zero integers, a and b.

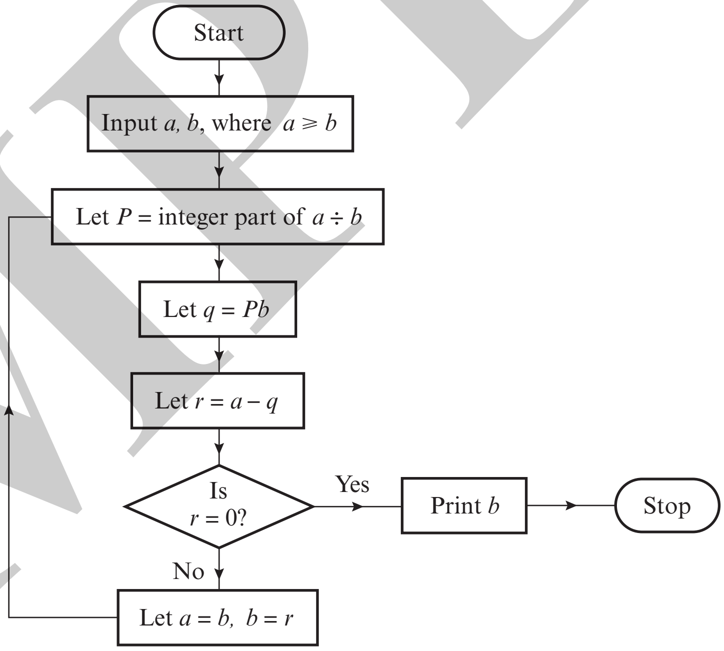
a Apply Euclid's algorithm to:

i 507 and 52 **(2 marks)**

ii 884 and 85 **(2 marks)**

iii 4845 and 3795 **(2 marks)**

b Explain what Euclid's algorithm does. **(2 marks)**



E/P **5** The flow chart describes an algorithm.

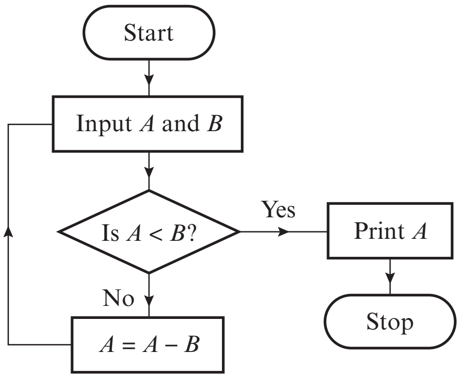
a Copy and complete this table, using the flow chart with $A = 18$ and $B = 7$.

A	B	$A < B$?	Output

(4 marks)

b Explain what is achieved by this flow chart. **(2 marks)**

c Given that $A = kB$ for some positive integer k, write down the output of the flow chart. **(1 mark)**



1.3 Bubble sort

A common data processing task is to sort an unordered list (a list which is not in order) into alphabetical or numerical order.

Lists can be put into ascending (increasing) or descending (decreasing) order.

- Unordered lists can be sorted using a **bubble sort** or a **quick sort**.
- In a bubble sort, we work through the list by comparing pairs of adjacent items (items that are next to each other) in the list.
  - If the items are in the correct order, leave them
  - If the items are not in the correct order, swap them

Once we have done this to all of the items in the list, we have completed the first pass.

If sorting the list into ascending order, the first pass will place the largest item in its correct position in the list.

If sorting the list into descending order, the first pass will place the smallest item in its correct position in the list.

We then repeat this until no swaps are made in a pass. If no swaps are made then the list is in order. You will need to write that no swaps have been made.

As the bubble sort develops, it is helpful to consider the original list as being divided into a **working list**, where comparisons are made, and a **sorted list** containing the items that are in their final positions. To start with, all items are in the working list.

This is the bubble sort algorithm:

1 Start at the beginning of the working list and move from left to right, comparing adjacent items.
 a If they are in order, leave them.
 b If they are not in order, swap them.

2 When you get to the end of the working list, the last item will be in its final position. This item is then no longer in the working list.

3 If you have made some swaps in the last pass, repeat step **1**.

4 When a pass is completed without any swaps, every item is in its final position and the list is in order.

Notation Each time you get to the end of the working list you complete one **pass** of the algorithm. The length of the working list reduces by 1 with each pass.

Notation The elements in the list 'bubble' to the end of the list in the same way that bubbles in a fizzy drink rise to the top. This is how the algorithm got its name.

You need to learn the bubble sort algorithm.

Example 7 SKILLS ANALYSIS

Use a bubble sort algorithm to arrange the list below into ascending order.

24 18 37 11 15 30

24	18	37	11	15	30	1st comparison: swap
18	24	37	11	15	30	2nd comparison: leave
18	24	37	11	15	30	3rd comparison: swap
18	24	11	37	15	30	4th comparison: swap
18	24	11	15	37	30	5th comparison: swap
18	24	11	15	30	37	End of first pass

After the second pass the list becomes

18 11 15 24 30 37

After the third pass the list is

11 15 18 24 30 37

After the fourth pass the list is

11 15 18 24 30 37

No swaps were made in the fourth pass, so the list is in order.

Hint In your exam you may be asked to show each comparison for one pass, but generally you will only be required to give the state of the list after each pass.

37 is already in its final position. It is now not in the working list. We now return to the start of the working list for the second pass.

After the third pass, the last three items are guaranteed to be in their final positions. In this example, the list is fully ordered but the algorithm requires another pass to be made.

Example 8 SKILLS REASONING/ARGUMENTATION

A list of n letters is to be sorted into alphabetical order, starting at the left-hand end of the list.

a Describe how to carry out the first pass of a bubble sort on the letters in the list.

b Carry out the first pass of a bubble sort to arrange the letters in the word ALGORITHM into alphabetical order, showing every step of the working.

c Show the order of the letters at the end of the second pass.

a Starting at the beginning of the list, compare the first two letters. If they are in alphabetical order, leave them in position, otherwise swap them. Continue through the list, to the end, comparing every pair of letters in the same way.

b	A	L	G	O	R	I	T	H	M	1st comparison: leave
	A	L	G	O	R	I	T	H	M	2nd comparison: swap
	A	G	L	O	R	I	T	H	M	3rd comparison: leave
	A	G	L	O	R	I	T	H	M	4th comparison: leave
	A	G	L	O	R	I	T	H	M	5th comparison: swap
	A	G	L	O	I	R	T	H	M	6th comparison: leave
	A	G	L	O	I	R	T	H	M	7th comparison: swap
	A	G	L	O	I	R	H	T	M	8th comparison: swap
	A	G	L	O	I	R	H	M	T	
c	A	G	L	I	O	H	M	R	T	

At the end of the first pass, the last letter is guaranteed to be in its correct place.

Example 9

SKILLS INNOVATION

Use a bubble sort to arrange these numbers into descending order.

39 57 72 39 17 24 48

39 57 72 39 17 24 48	$39 < 57$ so swap
57 39 72 39 17 24 48	$39 < 72$ so swap
57 72 39 39 17 24 48	$39 \nless 39$ so leave
57 72 39 39 17 24 48	$39 \nless 17$ so leave
57 72 39 39 17 24 48	$17 < 24$ so swap
57 72 39 39 24 17 48	$17 < 48$ so swap
57 72 39 39 24 48 17	
After 1st pass:	57 72 39 39 24 48 17
After 2nd pass:	72 57 39 39 48 24 17
After 3rd pass:	72 57 39 48 39 24 17
After 4th pass:	72 57 48 39 39 24 17
After 5th pass:	72 57 48 39 39 24 17

No swaps in 5th pass, so the list is in order.

Watch out Read the question carefully. You need to sort the list into **descending** order.

Note that the 48 is now between the two 39s. Do not treat the two 39s as one term.

Make sure that you make a statement like this to show that no swaps have been made and you have completed the algorithm.

Exercise 1C

SKILLS REASONING/ARGUMENTATION

1 Apply a bubble sort to arrange each list below into:

a ascending order **b** descending order

i	23	16	15	34	18	25	11	19
ii	N	E	T	W	O	R	K	S
iii	A5	D3	D2	A1	B4	C7	C2	B3

For each part, you need to show the state of the list only at the **end** of each pass.

Hint For part **iii,** order alphabetically then numerically. So C2 comes after A5 but before C7.

2 Perform a bubble sort to arrange these place names into alphabetical order.

Chester York Stafford Bridlington Burton Cranleigh Evesham

P **3** A list of n items is to be written in ascending order using a bubble sort.

a State the minimum number of passes needed.

b Describe the circumstances in which this number of passes would be sufficient.

c State the maximum number of passes needed.

d Describe the circumstances in which this number of passes would be needed.

E **4** Here is a list of exam scores:

63 48 57 55 32 48 72 49 61 39

The scores are to be put in order, highest first, using a bubble sort.

a Describe how to carry out the first pass. **(2 marks)**

b Apply a bubble sort to put the scores in the required order.
Only show the state of the list at the end of each pass. **(4 marks)**

1.4 Quick sort

The quick sort algorithm can be used to arrange a list into alphabetical or numerical order. In many cases, a quick sort is faster to perform than a bubble sort. We can thus say that it is more efficient.

In a quick sort, we choose an item which we call a **pivot**, and split the items into two sublists:

- One sublist contains items less than the pivot.
- The other sublist contains items greater than the pivot.

Hint If an item is equal to the pivot it can go in either sublist.

Once we have done this we have completed the first pass.

In doing the quick sort, the first pass will place the pivot item in its correct position in the list.

We then repeat this until all items are chosen as pivots, and then the list is in order. You will need to write that all items are chosen as pivots, which means that they are in order.

Here is the quick sort algorithm, used to sort a list into ascending order.

1 Choose the item at the midpoint of the list to be the first pivot.

2 Write down all the items that are less than the pivot, keeping their order, in a sublist.

3 Write down the pivot.

4 Write down the remaining items (those greater than the pivot) in a sublist.

5 Apply steps **1** to **4** to each sublist.

6 When all items have been chosen as pivots, stop.

Use the formula $\frac{(n+1)}{2}$ and round up, if needed, to find the midpoint of the list. For example, if there are 10 items in the list, $\frac{(10+1)}{2} = 5.5$ and so the 6th item in the list is the midpoint.

Do not sort the items as you write them down.

The number of pivots could double at each pass. There is 1 pivot at the first pass, there could be 2 at the second, 4 at the third, 8 at the fourth, and so on.

Example 10 SKILLS ANALYSIS

Use the quick sort algorithm to arrange the numbers below into ascending order.

21 24 42 29 23 13 8 39 38

21 24 42 29 23 13 8 39 38

21 13 8

21 13 8 $\boxed{23}$

21 13 8 $\boxed{23}$ 24 42 29 39 38

21 13 8 $\boxed{23}$ 24 42 29 39 38

8 $\boxed{13}$ 21 $\boxed{23}$ 24 $\boxed{29}$ 42 39 38

$\boxed{8}$ $\boxed{13}$ $\boxed{21}$ $\boxed{23}$ $\boxed{24}$ $\boxed{29}$ 38 $\boxed{39}$ 42

8 13 21 23 24 29 38 39 42

For n items, the pivot will be the $\frac{n+1}{2}$th item, rounding up if necessary.
There are 9 numbers in the list so the midpoint will be $\frac{9+1}{2} = 5$, so the pivot is the 5th number in the list. Circle it.

Write all the numbers less than 23.

Write the pivot in a box , then write the remaining numbers.

Now select a pivot in each sublist.

There are now four sublists so we choose four pivots (circled).

We can choose only two pivots this time. Each number has been chosen as a pivot, so the list is in order.

Example 11 SKILLS PROBLEM-SOLVING

Use the quick sort algorithm to arrange the list below into descending order.

37 20 17 26 44 41 27 28 50 17

37 20 17 26 44 41 27 28 50 17

44 50 $\boxed{41}$ 37 20 17 26 27 28 17

$\boxed{50}$ 44 $\boxed{41}$ 37 27 28 $\boxed{26}$ 20 17 17

$\boxed{50}$ $\boxed{44}$ $\boxed{41}$ 37 28 $\boxed{27}$ $\boxed{26}$ 20 17 $\boxed{17}$

$\boxed{50}$ $\boxed{44}$ $\boxed{41}$ 37 $\boxed{28}$ $\boxed{27}$ $\boxed{26}$ 20 $\boxed{17}$ $\boxed{17}$

50 44 41 37 28 27 26 20 17 17

There are 10 items in the list so the midpoint will be $\frac{(10+1)}{2} = 5.5$, and so the pivot is the 6th number in the list. Circle it.
Numbers greater than the pivot are to the left of the pivot, those smaller than the pivot are to the right, keeping the numbers in order. Numbers equal to the pivot may go either side, but must be dealt with in the same way each time you do a pass.

Two pivots are chosen, one for each sublist.

Now three pivots are selected.

We now choose the next two pivots, even if the sublist is in order.

The final pivots are chosen to give the list in order.

Watch out Colour is used here to make the method clear, but colours should not be used in your exam.

Exercise 1D

SKILLS REASONING/ARGUMENTATION

1 Use a bubble sort to arrange the list of numbers below into:

a ascending order **b** descending order

8 3 4 6 5 7 2

2 Use the quick sort algorithm to arrange the list below into:

a ascending order **b** descending order

22 17 25 30 11 18 20 14 7 29

3 Sort the letters below into alphabetical order using:

a a bubble sort **b** a quick sort

N H R K S C J E M P L

4 The list shows the test results of a group of students.

Alex	33	Hetal	9
Alison	56	Janelle	89
Amy	93	Josh	37
Annie	51	Lucy	57
Dewei	77	Masingur	19
Greg	91	Sam	29
Harry	49	Sophie	77

Produce a list of students, in descending order of their marks, using:

a a bubble sort **b** a quick sort

E/P **5** A list of n items is to be written in ascending order using the bubble sort algorithm.

a Find an expression, in terms of n, for the maximum number of comparisons to be made. **(2 marks)**

b Describe a situation where a bubble sort might be quicker than a quick sort. **(2 marks)**

c Decide whether a bubble sort or a quick sort will be quicker in the following cases:

i 1 2 3 7 4 5 6

ii 2 3 4 5 6 7 1

Explain how you made your decisions. **(4 marks)**

E **6** The table shows a list of nine names of students in a dance class.

Hassler	Sauver	Finch	Giannini	Mellor	Clopton	Miranti	Worth	Argi
H	S	F	G	Me	C	Mi	W	A

a Explain how to carry out the first pass of a quick sort algorithm to order the list alphabetically. **(2 marks)**

b Carry out the first two passes of a quick sort on this list, writing down the pivots used in each pass. **(3 marks)**

Challenge

SKILLS INNOVATION

You will need a pack of ordinary playing cards, with any jokers removed.
A pack of playing cards has 52 cards, split into 4 suits:

Hearts ♥ Diamonds ♦ Clubs ♣ Spades ♠

There are 13 cards in each suit, as follows:
Ace (1), 2, 3, 4, 5, 6, 7, 8, 9, 10, Jack (11), Queen (12), King (13)

a Use the quick sort algorithm to sort the cards into ascending order, from Ace to King within each suit and with the suits in the order: Hearts, Clubs, Diamonds, Spades. Follow these steps:

1. Shuffle the pack thoroughly and hold it face up.
2. Remove the 27th card and place it face up. This is your pivot card.
3. Go through the pack from the top. Place the cards into two piles depending on whether they are lower or higher than the pivot card.
4. Repeat these steps with each new pile, choosing the card halfway through the pile as the pivot card.

Record the total number of passes needed to sort the deck completely.

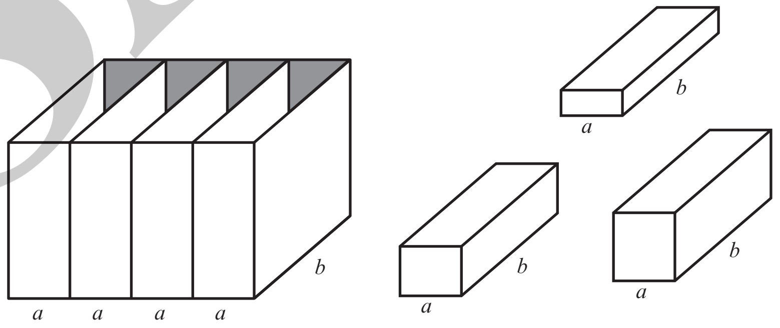
b Once the cards are in order, what single change could be made so that a bubble sort would require 51 passes to put the cards back in order?

Hint The final order should be:
A♥, 2♥, …, K♥, A♣, 2♣, …, K♣, A♦, …, K♦, A♠, …, K♠

1.5 Bin-packing algorithms

Bin packing refers to a whole class of problems.

The easiest way of thinking about this algorithm is to imagine boxes of fixed width a and length b, but varying heights, and stacking them into bins of width a and length b, using the minimum number of bins.



Similar problems are: loading cars of different lengths onto a ferry with several lanes of equal length, a plumber needing to cut sections from lengths of copper pipe, or recording music tracks onto a set of CDs.

You need to be able to apply three different **bin-packing** algorithms, and be aware of their strengths and weaknesses.

- The three bin-packing algorithms are: **first-fit**, **first-fit decreasing** and **full-bin**.

It is useful to first find a **lower bound** for the number of bins needed. There is no guarantee that you will be able to pack the items into this number of bins, but it will tell you if you have found an optimal solution.

Notation An **optimal solution** is one that cannot be improved upon. For bin packing, an optimal solution will use the smallest possible number of bins.

Example 12 SKILLS ANALYSIS

Nine boxes of fixed cross-section have heights, in metres, as follows.

0.3 0.7 0.8 0.8 1.0 1.1 1.1 1.2 1.5

They are to be packed into bins with the same fixed cross-section and height 2 m. Determine the lower bound for the number of bins needed.

$0.3 + 0.7 + 0.8 + 0.8 + 1.0 + 1.1 + 1.1 + 1.2$
$+ 1.5 = 8.5\text{ m}$

$\frac{8.5}{2} = 4.25$ bins

So a minimum of 5 bins will be needed.

Sum the heights and divide by the bin size. You must always round **up** to determine the lower bound.

Watch out In practice, it may not be possible to pack these boxes into 5 bins. The lower bound simply tells us that **at least** 5 bins will be needed.

With small amounts of data it is often possible to ‘spot’ an optimal answer.

The algorithms you will learn in this chapter will not necessarily find an **optimal solution**, but can be applied quickly.

- The first-fit algorithm works by considering items in the order they are given.

First-fit algorithm

1 Take the items **in the order given**.
2 Place each item in the first available bin that can take it. Start from bin 1 each time.

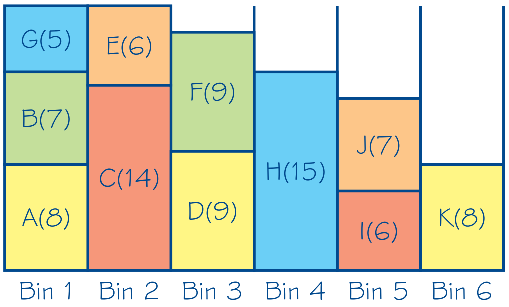
Advantage: It is quick to apply.
Disadvantage: It is not likely to lead to a good solution.

Online See the operation of the first-fit algorithm using GeoGebra.

Example 13 SKILLS EXECUTIVE FUNCTION

Use the first-fit algorithm to pack the following items into bins of size 20. (The numbers in brackets are the size of the item.) State the number of bins used and the amount of wasted space.

A(8) B(7) C(14) D(9) E(6) F(9) G(5) H(15) I(6) J(7) K(8)



This used 6 bins and there are
2 + 5 + 7 + 12 = 26 units of waste of space.

A(8) goes into bin 1, leaving space of 12.
B(7) goes into bin 1, leaving space of 5.
C(14) goes into bin 2, leaving space of 6.
D(9) goes into bin 3, leaving space of 11.
E(6) goes into bin 2, leaving space of 0.
F(9) goes into bin 3, leaving space of 2.
G(5) goes into bin 1, leaving space of 0.
H(15) goes into bin 4, leaving space of 5.
I(6) goes into bin 5, leaving space of 14.
J(7) goes into bin 5, leaving space of 7.
K(8) goes into bin 6, leaving space of 12.

- The first-fit decreasing algorithm requires the items to be in descending order before applying the algorithm.

First-fit decreasing algorithm

1 Sort the items so that they are in descending order.
2 Apply the first-fit algorithm to the reordered list.

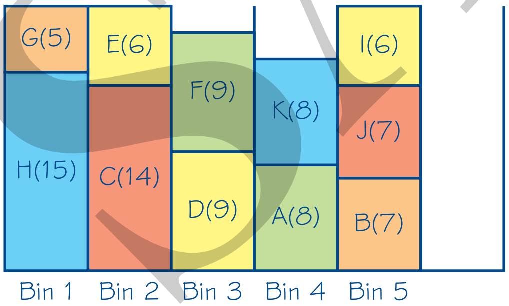
Advantages: You usually get a fairly good solution.
It is easy to apply.
Disadvantage: You may not get an optimal solution.

Online See the operation of the first-fit decreasing algorithm using GeoGebra.

Example 14 SKILLS ANALYSIS

Apply the first-fit decreasing algorithm to the data given in Example 13.

Sort the data into descending order:
H(15) C(14) D(9) F(9) A(8) K(8) B(7) J(7) E(6) I(6) G(5)



This used 5 bins and there are
2 + 4 = 6 units of wasted space.

H(15) goes into bin 1, leaving space of 5.
C(14) goes into bin 2, leaving space of 6.
D(9) goes into bin 3, leaving space of 11.
F(9) goes into bin 3, leaving space of 2.
A(8) goes into bin 4, leaving space of 12.
K(8) goes into bin 4, leaving space of 4.
B(7) goes into bin 5, leaving space of 13.
J(7) goes into bin 5, leaving space of 6.
E(6) goes into bin 2, leaving space of 0.
I(6) goes into bin 5, leaving space of 0.
G(5) goes into bin 1, leaving space of 0.

- Full-bin packing uses **inspection** to select items that will combine to fill bins. Remaining items are packed using the first-fit algorithm.

Full-bin packing

1. Use observation to find combinations of items that will fill a bin. Pack these items first.
2. Any remaining items are packed using the first-fit algorithm.

Advantage: You usually get a good solution.
Disadvantage: It is difficult to do, especially when the numbers are plentiful and awkward.

Example 15 SKILLS EXECUTIVE FUNCTION

A(8) B(7) C(10) D(11) E(13) F(17) G(4) H(6) I(12) J(14) K(9)

The items above are to be packed in bins of size 25.

a Determine the lower bound for the number of bins.

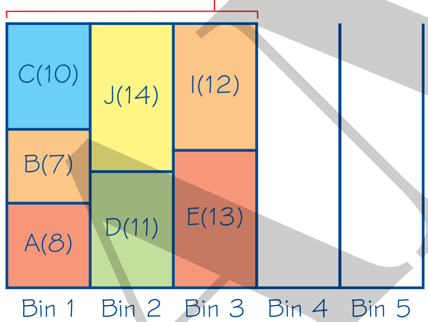
b Apply the full-bin algorithm.

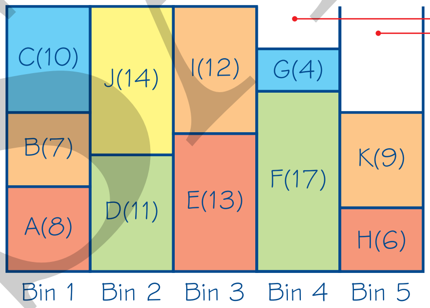
c Is your solution optimal? Give a reason for your answer.

a $111 \div 25 = 4.44$
So lower bound is 5 bins.

b Three groupings of numbers that sum to 25 can be made as follows:
$8 + 7 + 10 = 25$
$11 + 14 = 25$
$13 + 12 = 25$

The first three bins are full bins.





We now apply the first-fit algorithm to the remainder.
F(17) goes into bin 4, leaving space of 8.
G(4) goes into bin 4, leaving space of 4.
H(6) goes into bin 5, leaving space of 19.
K(9) goes into bin 5, leaving space of 10.

c The lower bound is 5, and 5 bins were used, so the solution is optimal.

Example 16 SKILLS EXECUTIVE FUNCTION

A plumber needs to cut the following lengths of copper pipe. (Lengths are in metres.)

A(0.8) B(0.8) C(1.4) D(1.1) E(1.3) F(0.9) G(0.8) H(0.9) I(0.8) J(0.9)

The pipe comes in lengths of 2.5 m.

a Calculate the lower bound of the number of lengths of pipe needed.

b Use the first-fit decreasing algorithm to determine how the required lengths may be cut from the 2.5 m lengths.

c Use full-bin packing to find an optimal solution.

a $\frac{0.8 + 0.8 + 1.4 + 1.1 + 1.3 + 0.9 + 0.8 + 0.9 + 0.8 + 0.9}{2.5}$

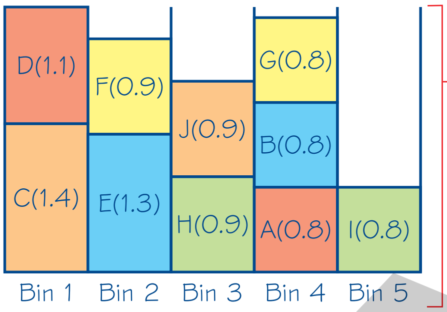
$= 3.88$

So at least 4 lengths are required.

b Sorting into descending order,

C(1.4), E(1.3), D(1.1), F(0.9), H(0.9), J(0.9), A(0.8), B(0.8), G(0.8), I(0.8)

Since a sort was not asked for, this can be done by inspection.



C goes into bin 1, leaving space of 1.1.
E goes into bin 2, leaving space of 1.2.
D goes into bin 1, leaving space of 0.
F goes into bin 2, leaving space of 0.3.
H goes into bin 3, leaving space of 1.6.
J goes into bin 3, leaving space of 0.7.
A goes into bin 4, leaving space of 1.7.
B goes into bin 4, leaving space of 0.9.
G goes into bin 4, leaving space of 0.1.
I goes into bin 5, leaving space of 1.7.

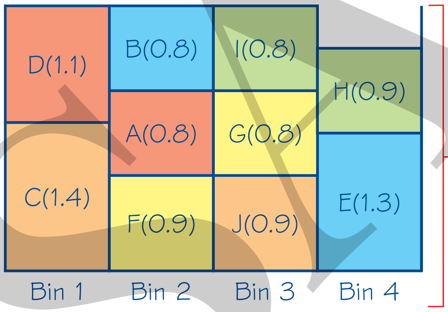
c By inspection,

C(1.4) + D(1.1) = 2.5

F(0.9) + A(0.8) + B(0.8) = 2.5

J(0.9) + G(0.8) + I(0.8) = 2.5

A full-bin solution is:



In part **a** we found that at least 4 lengths would be needed, so this solution is optimal since it uses 4 lengths.

Exercise 1E

SKILLS REASONING/ARGUMENTATION

1 18 4 23 8 27 19 3 26 30 35 32

The above items are to be packed in bins of size 50.

a Calculate the lower bound for the number of bins.

b Pack the items into the bins using:
i the first-fit algorithm **ii** the first-fit decreasing algorithm **iii** the full-bin algorithm

2 Laura hosts an internet music channel and wishes to play the 13 pieces of music listed below. Each day, she hosts a session which is at most 3 hours long.

Piece of music	A	B	C	D	E	F	G	H	I	J	K	L	M
Length (minutes)	30	30	30	45	45	60	60	60	60	75	90	120	120

a Apply the first-fit algorithm, in the order A to M, to determine the number of days that need to be used. State which music is played on each day.

b Repeat part **a** using the first-fit decreasing algorithm.

c Is your answer to part **b** optimal? Give a reason for your answer.

Laura finds that her session time is now reduced to only 2 hours.

d Use the full-bin algorithm to determine the number of days she needs to use. State which music is played on each day.

E **3** A small ferry loads vehicles into 30 m lanes. The vehicles are loaded bumper to bumper.

	Vehicle	Length (m)
A	car	4 m
B	car and trailer	7 m
C	lorry	13 m
D	van	6 m
E	lorry	13 m

	Vehicle	Length (m)
F	car	4 m
G	lorry	12 m
H	lorry	14 m
I	van	6 m
J	lorry	11 m

a Describe one difference between the first-fit and full-bin methods of bin packing. **(1 mark)**

b Use the first-fit algorithm to determine the number of lanes needed to load all the vehicles onto the ferry. **(4 marks)**

c Use a full-bin method to obtain an optimal solution using the minimum number of lanes. Explain why your solution is optimal. **(4 marks)**

E **4** The ground floor of an office block is to be fully recarpeted, with specially made carpet incorporating the firm's logo. The carpet comes in rolls of 15 m.

The following lengths are required.

A 3 m	D 4 m	G 5 m	J 7 m
B 3 m	E 4 m	H 5 m	K 8 m
C 4 m	F 4 m	I 5 m	L 8 m

The lengths are arranged in **ascending** order of size.

a Obtain a lower bound for the number of rolls of carpet needed. **(2 marks)**

b Use the first-fit decreasing bin-packing algorithm to determine the number of rolls needed. State the length of carpet that is wasted using this method. **(3 marks)**

c Give one disadvantage of the first-fit decreasing bin-packing algorithm. **(1 mark)**

d Use a full-bin method to obtain an optimal solution, and state the total length of wasted carpet using this method. **(4 marks)**

E/P **5** Eight computer programs need to be copied onto 40 GB USB sticks. The size of each program is given below.

Program	A	B	C	D	E	F	G	H
Size (GB)	8	16	17	21	22	24	25	25

a Use the first-fit decreasing algorithm to determine which programs should be recorded onto each USB stick. **(3 marks)**

b Calculate a lower bound for the number of USB sticks needed. **(2 marks)**

c Explain why it is not possible to record these programs on the number of USB sticks found in part **b**. **(1 mark)**

Problem-solving

Consider the programs over 20 GB in size.

1.6 Binary search

You need to be able to carry out a binary search.

A binary search will look through an **ordered** list to find out whether or not an item you are trying to find is in the list. If the item is in the list, the binary search will locate its position within the list.

If the list is not in order, then you may need to use a bubble sort or quick sort to put the items into order first.

- In a binary search, we look at halving the size of the list each time we perform a **pass**.
  - In a binary search, we locate the midpoint of the list using $\frac{n+1}{2}$. We call this the **pivot**. Like with the quick sort, we round this up if it is not an integer.
  - We compare this midpoint with the item we are trying to locate; this will help us decide which half of the list to choose.
  - Eventually we will get to one item – it will either be the item we are trying to locate, or it will not be. In this case we can say that the item we were trying to locate is not in the list.

Here is the binary search algorithm to locate an item in a list:

1 Select the midpoint of the list using $\frac{n+1}{2}$ and round up if necessary. This is the pivot.

2 **a** If the pivot is the item we are locating, then the search is complete.

b If the pivot is after the item we are locating, then we look in the first half of the list.

c If the pivot is before the item we are locating, then we look in the second half of the list.

3 Repeat steps 1 and 2 to each remaining list until the item is located. If the item is not found, then it is not in the list.

Example 17

Use the binary search algorithm to try to locate these names in the list that follows.

a Robinson **b** Davies

1	Acharya	6	Laing
2	Blackstock	7	Leung
3	Cheung	8	Robinson
4	Coetzee	9	Saludo
5	Fowler	10	Xiao

Watch out Remember that a search can be unsuccessful. You may be asked to try to locate something that is not in the list. You must be able to show that the item is not in the list.

a The middle name is the $\left(\frac{n+1}{2} = 5.5\right)$ 6th name:
6 Laing

Remember to round up if $\frac{n+1}{2}$ is not an integer.

Robinson is after Laing, so the list reduces to
1 Leung
2 Robinson
3 Saludo
4 Xiao

Since Robinson is after Laing, Robinson cannot be in the first part of the list and so we consider the list after the pivot.

The middle name in this sublist is the $\left(\frac{4+1}{2} = 2.5\right)$ 3rd name:
3 Saludo

Robinson is before Saludo so it cannot be in the second list and so we consider the list before the pivot.

Robinson is before Saludo, so the list reduces to:
1 Leung
2 Robinson

The middle name in this sublist is the $\left(\frac{2+1}{2} = 1.5\right)$ 2nd name:
2 Robinson

The search is complete.
Robinson has been found in the list.

It is important to write this down.

b The middle name is the $\left(\frac{10+1}{2} = 5.5\right)$ 6th name:
6 Laing

Consider the list **before** the pivot.

Davies is before Laing so the list reduces to:
1 Acharya
2 Blackstock
3 Cheung
4 Coetzee
5 Fowler

The middle name is the $\left(\frac{5+1}{2} = 3\right)$ 3rd name:
3 Cheung

Consider the list **after** the pivot.

Davies is after Cheung so the list reduces to
1 Coetzee
2 Fowler

The middle name is the $\left(\frac{n+1}{2} = 1.5\right)$ 2nd name:
2 Fowler
Davies is before Fowler so the list reduces to:
1 Coetzee
The list has only one item which is not Davies.
Therefore Davies is not in the list.

Consider the list **before** the pivot.

It is important to write this down.

Exercise 1F SKILLS ANALYSIS

1 Use the binary search algorithm to try to locate these names in the list that follows:

a Connock **b** Walkey **c** Peabody

1 Berry
2 Connock
3 Li
4 Sully
5 Tapner
6 Walkey
7 Wilson
8 Wu

2 Use the binary search algorithm to try to locate these numbers in the list that follows:

a 21 **b** 5

1 3	**3** 7	**5** 10	**7** 15	**9** 18	**11** 21
2 4	**4** 9	**6** 13	**8** 17	**10** 20	**12** 24

(P) **3** The binary search algorithm is applied to an ordered list of n items.
Find the maximum number of times the algorithm is run when n is equal to:

a 100 **b** 1000 **c** 10 000

4 a Use the quick sort algorithm to put the list below into ascending order.

1 Adam	**6** Ramin	**11** Oli	**16** Miranda
2 Ed	**7** Alex	**12** Lotus	**17** Matt
3 Lei	**8** Emily	**13** Des	**18** Katie
4 Lottie	**9** Felix	**14** George	**19** Doug
5 Saul	**10** Leo	**15** Jess	**20** Hongmei

b Use the binary search algorithm to try to locate:

i George **ii** David **iii** Jess

Chapter review 1

(E) **1** Use the bubble-sort algorithm to sort, in ascending order, the list

27 15 2 38 16 1

giving the state of the list at each stage. **(4 marks)**

E/P **2** **a** Use the bubble-sort algorithm to sort, in descending order, the list

25 42 31 22 26 41

giving the state of the list on each occasion when two or more values are interchanged (swapped). **(4 marks)**

b Find the **maximum** number of interchanges needed to sort a list of six pieces of data using the bubble-sort algorithm. **(2 marks)**

E **3** 8 4 13 2 17 9 15

This list of numbers is to be sorted into ascending order.
Perform a quick sort to obtain the sorted list, giving the state of the list after each rearrangement. **(5 marks)**

E **4** 111 103 77 81 98 68 82 115 93

a The list of numbers above is to be sorted into descending order. Perform a quick sort to obtain the sorted list, giving the state of the list after each rearrangement and indicating the pivot elements used. **(5 marks)**

b **i** Use the first-fit decreasing bin-packing algorithm to fit the data into bins of size 200. **(3 marks)**

ii Explain how you decided in which bin to place the number 77. **(1 mark)**

E **5** Trishna wishes to record eight television programmes. The lengths of the programmes, in minutes, are:

75 100 52 92 30 84 42 60

Trishna decides to use 2-hour (120 minute) DVDs only to record all of these programmes.

a Explain how to apply the first-fit decreasing bin-packing algorithm. **(2 marks)**

b Use this algorithm to fit these programmes onto the smallest number of DVDs possible, stating the total amount of unused space on the DVDs. **(3 marks)**

Trishna wants to record an additional two 25-minute programmes.

c Determine whether she can do this using only 5 DVDs, giving reasons for your answer. **(3 marks)**

E **6** A DIY enthusiast requires the following 14 pieces of wood as shown in the table.

Length in metres	0.4	0.6	1	1.2	1.4	1.6
Number of pieces	3	4	3	2	1	1

The DIY store sells wood in 2 m and 2.4 m lengths. He considers buying six 2 m lengths of wood.

a Explain why he will not be able to cut all of the lengths he requires from these six 2 m lengths. **(2 marks)**

b He eventually decides to buy 2.4 m lengths. Use a first-fit decreasing bin-packing algorithm to show how he could use six 2.4 m lengths to obtain the pieces he requires. **(4 marks)**

c Obtain a solution that requires only five 2.4 m lengths. **(4 marks)**

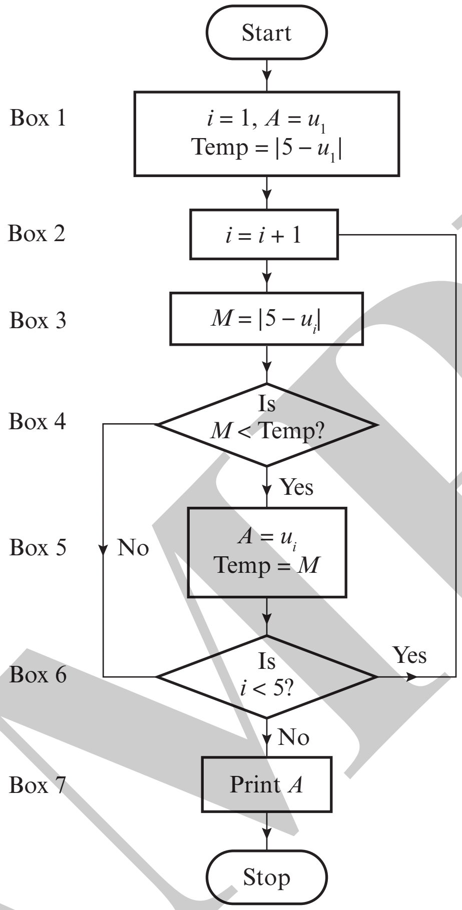
E/P **7** The algorithm described by the flow chart below is to be applied to the five pieces of data below.

$u_1 = 6.1,\ u_2 = 6.9,\ u_3 = 5.7,\ u_4 = 4.8,\ u_5 = 5.3$

Hint This question uses the modulus function. If $x \neq y$, $|x - y|$ is the positive difference between x and y, e.g. $|5 - 6.1| = 1.1$.

a Obtain the final output of the algorithm using the five values given for u_1 to u_5. **(4 marks)**

b In general, for any set of values u_1 to u_5, explain what the algorithm achieves. **(2 marks)**



c If Box 4 in the flow chart is altered to 'Is M > Temp?' state what the algorithm achieves now. **(1 mark)**

E **8** A plumber is cutting lengths of PVC pipe for a bathroom. The lengths needed, in metres, are:

0.3 2.0 1.3 1.6 0.3 1.3 0.2 0.1 2.0 0.5

The pipe is sold in 2 m lengths.

a Carry out a bubble sort to produce a list of the lengths needed in **descending** order. Give the state of the list after each pass. **(4 marks)**

b Apply the first-fit decreasing bin-packing algorithm to your ordered list to determine the total number of 2 m lengths of pipe needed. **(3 marks)**

c Does the answer to part **b** use the minimum number of 2 m lengths? You must justify your answer. **(2 marks)**

E/P **9** Here are the names of eight students in an A level group:

Manisha, Vivien, Cath, Alex, Da Ming, Beth, Kandis, Sze-To

Use a quick sort to put the names in alphabetical order. Show the result of each pass and identify the pivots. **(5 marks)**

Challenge

E/P **10** A binary search is to be performed on a list of names to try to locate Kim.

1	Jenny	**6**	Hyo
2	Merry	**7**	Kim
3	Charles	**8**	Richard
4	Ben	**9**	Greg
5	Toby	**10**	Freya

a Explain why a binary search cannot be performed with the list in its present form. **(1 mark)**

b Using an appropriate algorithm, alter the list so that a binary search can be performed, showing the state of the list after each complete iteration. State the name of the algorithm you have used. **(4 marks)**

c Use the binary search algorithm to locate the name Kim in the list you obtained in **b**. You must make your method clear. **(4 marks)**

Summary of key points

1 An **algorithm** is a finite sequence of step-by-step instructions carried out to solve a problem.

2 In a **flow chart**, the shape of each box tells you about its function.

3 Unordered lists can be sorted using a bubble sort or a quick sort.

4 In a **bubble sort**, you compare adjacent items in a list:
- If they are in order, leave them.
- If they are not in order, swap them.
- The list is in order when a pass is completed without any swaps.

5 In a **quick sort**, you select a pivot and then split the items into two sublists:
- One sublist contains items less than the pivot.
- The other sublist contains items greater than the pivot.
- You then select further pivots from within each sublist and repeat the process.

6 The three bin-packing algorithms are first-fit, first-fit decreasing, and full-bin:
- The **first-fit** algorithm works by considering items in the order they are given.
- The **first-fit decreasing** algorithm requires the items to be in descending order before applying the algorithm.
- **Full-bin packing** uses inspection to select items that combine to fill bins completely. Remaining items are packed using the first-fit algorithm.

7 The three bin-packing algorithms have the following advantages and disadvantages:

Type of algorithm	Advantage	Disadvantage
First-fit	Quick to apply	Not likely to lead to a good solution
First-fit decreasing	Usually a good solution; easy to apply	May not get an optimal solution
Full-bin	Usually a good solution	Difficult to do, especially when the numbers are plentiful or awkward

8 A **binary search** will search an ordered list to find out whether an item is in the list. If it is in the list, it will locate its position in the list.
In a binary search, the pivot is the middle item of the list. If the target item is not the pivot, the pivot and half of the list are discarded. The list length halves at each pass.
The middle of n items is found by $\frac{n+1}{2}$, rounding up if necessary.

Exercise 1A

1 a

A	B
~~244~~	~~125~~
~~122~~	~~250~~
61	500
~~30~~	~~1000~~
15	2000
7	4000
3	8000
1	16 000
Total	30 500

b

A	B
125	244
~~62~~	~~488~~
31	976
15	1952
7	3904
3	7808
1	15 616
Total	30 500

c

A	B
~~256~~	~~123~~
~~128~~	~~246~~
~~64~~	~~492~~
~~32~~	~~984~~
~~16~~	~~1968~~
~~8~~	~~3936~~
~~4~~	~~7872~~
~~2~~	~~15 744~~
1	31 488
Total	31 488

2 a 1 $\frac{a}{b}=\frac{9}{4}\ \frac{c}{d}=\frac{4}{3}\ a=9,\ b=4,\ c=4,\ d=3$

2 $e=ad=9\times 3=27$

3 $f=bc=4\times 4=16$

4 answer is $\frac{27}{16}$

b It divides the first fraction by the second fraction.

3 a

Instruction step	n	A	Print
1	1	1	
2			1
3		4	
4	2		
5	$2\leqslant 10$ go to step 2		
2			4
3		9	
4	3		
5	$3\leqslant 10$ go to step 2		
2			9
3		16	
4	4		
5	$4\leqslant 10$ go to step 2		
2			16
3		25	
4	5		
5	$5\leqslant 10$ go to step 2		
2			25
3		36	
4	6		
5	$6\leqslant 10$ go to step 2		
2			36
3		49	
4	7		
5	$7\leqslant 10$ go to step 2		
2			49
3		64	
4	8		
5	$8\leqslant 10$ go to step 2		
2			64
3		81	
4	9		
5	$9\leqslant 10$ go to step 2		
2			81
3		100	
4	10		
5	$10\leqslant 10$ go to step 2		
2			100
3		121	
4	11		
5	$11\leqslant 10$ continue to step 6		
6	Stop		

Output 1, 4, 9, 16, 25, 36, 49, 64, 81, 100

b The algorithm produces the squares of the first 10 natural numbers.

4 a i

Step	A	r	c	$\lvert r-c \rvert$	s	**Print *r***
1	253	12				
2			21.083			
3				9.083		
4					16.542	
5		16.542				
$6 \to 2$			15.294			
3				1.248		
4					15.918	
5		15.918				
$6 \to 2$			15.894			
3				0.024		
4					15.906	
5		15.906				
$6 \to 2$			15.906			
$3 \to 7$				0		
7						$r = 15.906$
8 stop						

ii

Step	A	r	c	$\lvert r-c \rvert$	s	**Print *r***
1	79	10				
2			7.900			
3				2.1		
4					8.950	
5		8.95				
$6 \to 2$			8.827			
3				0.123		
4					8.889	
5		8.889				
$6 \to 2$			8.887			
$3 \to 7$				0.002		
7						Print 8.889

4 a iii

Step	A	r	c	$\lvert r-c \rvert$	s	**Print *r***
1	4275	50				
2			85.500			
3				35.5		
4					67.750	
5		67.75				
$6 \rightarrow 2$			63.100			
3				4.65		
4					65.425	
5		65.425				
$6 \rightarrow 2$			65.342			
3				0.083		
4					65.384	
5		65.384				
$6 \rightarrow 2$			65.383			
$3 \rightarrow 7$				0.001		
7						Print 65.384

b Finds the square root of A correct to 1 decimal place.

Exercise 1B

1 a

a	b	c	d	$d < 0?$	$d = 0?$	x
4	−12	9	0	No	Yes	1.5

Equal roots are $x = 1.5$.

b

a	b	c	d	$d < 0?$	$d = 0?$	x_1	x_2
−6	13	5	289	No	No	$-\frac{1}{3}$	$\frac{5}{2}$

Roots are $-\frac{1}{3}$ and $\frac{5}{2}$

c

a	b	c	d	$d < 0?$
3	−8	11	−68	Yes

No real roots.

2 a i

	n	A	T	$T < A?$	$n < 5?$
box 1	1	28			
box 2	2				
box 3			26		
box 4				Yes	
box 5		26			
box 6					Yes
box 2	3				
box 3			23		
box 4				Yes	
box 5		23			
box 6					Yes
box 2	4				
box 3			25		
box 4				No	
box 6					Yes
box 2	5				
box 3			21		
box 4				Yes	
box 5		21			
box 6					No
box 7	Output is 21				

2 a ii

	n	A	T	$T < A$?	$n < 5$?
box 1	1	11			
box 2	2				
box 3			8		
box 4				Yes	
box 5		8			
box 6					Yes
box 2	3				
box 3			9		
box 4				No	
box 6					Yes
box 2	4				
box 3			8		
box 4				No	
box 6					Yes
box 2	5				
box 3			5		
box 4				Yes	
box 5		5			
box 6					No
box 7	Output is 5				

b It will find the largest number in the list.

c box 6 – changed to 'Is $n < 8$?'

3 a

a	b	$\lvert a-b\rvert$	$\lvert a-b\rvert < 0.001$?
2	1.7652	0.2348	No
1.7652	1.8112	0.046	No
1.8112	1.8029	0.0083	No
1.8029	1.8044	0.0015	No
1.8044	1.8041	0.0003	Yes

Output 1.8041 (4 d.p.)

3 b

a	b	$\lvert a-b \rvert$	$\lvert a-b \rvert < 0.001$?
20	–5.7740	25.774	No
–5.7740	–2.0931	3.6809	No
–2.0931	1.7446	3.8377	No
1.7446	1.8149	0.0703	No
1.8149	1.8022	0.0127	No
1.8022	1.8045	0.0023	No
1.8045	1.8041	0.0004	Yes

Output 1.8041 (4 d.p.)
The sequence produced in part **b** is initially quite different to the sequence produced in part **a** but both sequences converge to the same root.

4 a i

a	b	p	q	r	$r = 0$?
507	52				
		9			
			468		
				39	
					No
52	39				
		1			
			39		
				13	
					No
39	13				
		3			
			39		
				0	
					Yes

Print 13

4 a ii

a	b	p	q	r	$r = 0$?
884	85				
		10			
			850		
				34	
					No
85	34				
		2			
			68		
				17	
					No
34	17				
		2			
			34		
				0	
					Yes

Print 17

4 a iii

a	b	p	q	r	$r = 0$?
4845	3795				
		1			
			3795		
				1050	
					No
3795	1050				
		3			
			3150		
				645	
					No
1050	645				
		1			
			645		
				405	
					No
645	405				
		1			
			405		
				240	
					No
405	240				
		1			
			240		
				165	
					No
240	165				
		1			
			165		
				75	
					No
165	75				
		2			
			150		
				15	
					No
75	15				
		5			
			75		
				0	
					Yes

Print 15

b Euclid's algorithm finds the HCF.

5 a

A	B	$A < B$?	**Output**
18	7	No	
11	7	No	
4	7	Yes	4

5 **b** The flow diagram calculates the remainder when A is divisible by B.

c The output is 0, so A is completely divisible by B for some positive integer k.

Exercise 1C

1 i a Bubbling left to right
23 16 15 34 18 25 11 19
23 > 16 so swap
16 23 15 34 18 25 11 19
23 > 15 so swap
16 15 23 34 18 25 11 19
23 < 34 so leave
16 15 23 34 18 25 11 19
34 > 18 so swap
16 15 23 18 34 25 11 19
34 > 25 so swap
16 15 23 18 25 34 11 19
34 > 11 so swap
16 15 23 18 25 11 34 19
34 > 19 so swap
16 15 23 18 25 11 19 34
After 1st pass: 16 15 23 18 25 11 19 34
After 2nd pass: 15 16 18 23 11 19 25 34
After 3rd pass: 15 16 18 11 19 23 25 34
After 4th pass: 15 16 11 18 19 23 25 34
After 5th pass: 15 11 16 18 19 23 25 34
After 6th pass: 11 15 16 18 19 23 25 34
After 7th pass: 11 15 16 18 19 23 25 34
No swap in 7th pass, so the list is in order.

b Bubbling left to right
23 16 15 34 18 25 11 19
23 > 16 so leave
23 16 15 34 18 25 11 19
16 > 15 so leave
23 16 15 34 18 25 11 19
15 < 34 so swap
23 16 34 15 18 25 11 19
15 < 18 so swap
23 16 34 18 15 25 11 19
15 < 25 so swap
23 16 34 18 25 15 11 19
15 > 11 so leave
23 16 34 18 25 15 11 19
11 < 19 so swap
23 16 34 18 25 15 19 11
After 1st pass: 23 16 34 18 25 15 19 11
After 2nd pass: 23 34 18 25 16 19 15 11
After 3rd pass: 34 23 25 18 19 16 15 11
After 4th pass: 34 25 23 19 18 16 15 11
After 5th pass: 34 25 23 19 18 16 15 11
No swap in 5th pass, so the list is in order.

ii a After 1st pass: E N T O R K S W
After 2nd pass: E N O R K S T W
After 3rd pass: E N O K R S T W
After 4th pass: E N K O R S T W
After 5th pass: E K N O R S T W
After 6th pass: E K N O R S T W
No swap in 6th pass, so the list is in order.

1 ii b After 1st pass: N T W O R K S E
After 2nd pass: T W O R N S K E
After 3rd pass: W T R O S N K E
After 4th pass: W T R S O N K E
After 5th pass: W T S R O N K E
After 6th pass: W T S R O N K E
No swap in 6th pass, so the list is in order.

iii a After 1st pass: A5 D2 A1 B4 C7 C2 B3 D3
After 2nd pass:A5 A1 B4 C7 C2 B3 D2D3
After 3rd pass:A1 A5 B4 C2 B3 C7 D2 D3
After 4th pass:A1 A5 B4 B3 C2 C7 D2 D3
After 5th pass:A1 A5 B3 B4 C2 C7 D2 D3
After 6th pass:A1 A5 B3 B4 C2 C7 D2 D3
No swap in 6th pass, so the list is in order.

b After 1st pass: D3 D2 A5 B4 C7 C2 B3 A1
After 2nd pass:D3 D2 B4 C7 C2 B3 A5A1
After 3rd pass:D3 D2 C7 C2 B4 B3 A5 A1
After 4th pass:D3 D2 C7 C2 B4 B3 A5 A1
No swap in 4th pass, so the list is in order.

2 Bubbling left to right
After 1st pass: Ch St Br Bu Cr Ev Yo
After 2nd pass: Ch Br Bu Cr Ev St Yo
After 3rd pass:Br Bu Ch Cr Ev St Yo
After 4th pass:Br Bu Ch Cr Ev St Yo

No swap in 4th pass, so the list is in order.
Bridlington, Burton, Chester, Cranleigh, Evesham, Stafford, York

3 a If there are n items, then there are $n - 1$ pairs of items that need to be compared. For a pair, the minimum number of passes needed is 1.

b One pass is sufficient if the items are already in ascending order.

c The maximum number of passes needed is n

d n passes are needed if the smallest item is at the end of the list.

4 a Bubble left to right of the list comparing each pair of numbers. If the first number of a pair is greater than or equal to the second, make no change. If the first number of a pair is less than the second, swap them. This is the first pass.

4 b After 1st pass: 63 57 55 48 48 72 49 61 39 32
After 2nd pass: 63 57 55 48 72 49 61 48 39 32
After 3rd pass: 63 57 55 72 49 61 48 48 39 32
After 4th pass: 63 57 72 55 61 49 48 48 39 32
After 5th pass: 63 72 57 61 55 49 48 48 39 32
After 6th pass: 72 63 61 57 55 49 48 48 39 32
After 7th pass: 72 63 61 57 55 49 48 48 39 32
No swap in 7th pass, so the scores are in order.

Exercise 1D

1 a 8 3 4 6 5 7 2

Bubbling left to right

1st pass	3	4	6	5	7	2	8
2nd pass	3	4	5	6	2	7	8
3rd pass	3	4	5	2	6	7	8
4th pass	3	4	2	5	6	7	8
5th pass	3	2	4	5	6	7	8
6th pass	2	3	4	5	6	7	8

b Bubbling left to right

1st pass	8	4	6	5	7	3	2
2nd pass	8	6	5	7	4	3	2
3rd pass	8	6	7	5	4	3	2
4th pass	8	7	6	5	4	3	2

2 a

22	17	25	30	11	18	20	14	7	29
17	11	14	7	$\boxed{18}$	22	25	30	20	29
11	7	$\boxed{14}$	17		22	25	20	29	$\boxed{30}$
$\boxed{7}$	11		$\boxed{17}$		$\boxed{20}$	22	25	29	
	$\boxed{11}$					22	$\boxed{25}$	29	
						$\boxed{22}$		$\boxed{29}$	
7	11	14	17	18	20	22	25	29	30

b

22	17	25	30	11	18	20	14	7	29
22	25	30	20	29	$\boxed{18}$	17	11	14	7
$\boxed{30}$	22	25	20	29		17	$\boxed{14}$	11	7
	29	$\boxed{25}$	22	20		$\boxed{17}$		11	$\boxed{7}$
	$\boxed{29}$		22	$\boxed{20}$				$\boxed{11}$	
			$\boxed{22}$						
30	29	25	22	20	18	17	14	11	7

3 a

N	H	R	K	S	C	J	E	M	P	L

Bubbling from left to right.

H	N	K	R	C	J	E	M	P	L	S
H	K	N	C	J	E	M	P	L	R	S
H	K	C	J	E	M	N	L	P	R	S
H	C	J	E	K	M	L	N	P	R	S
C	H	E	J	K	L	M	N	P	R	S
C	E	H	J	K	L	M	N	P	R	S

b

N	H	R	K	S	C	J	E	M	P	L
$\boxed{C}$	N	H	R	K	S	J	E	M	P	L
	H	E	$\boxed{J}$	N	R	K	S	M	P	L
	$\boxed{E}$	H		N	R	K	M	P	L	$\boxed{S}$
		$\boxed{H}$		K	L	$\boxed{M}$	N	R	P	
				K	$\boxed{L}$		N	P	$\boxed{R}$	
				$\boxed{K}$			N	$\boxed{P}$		
							$\boxed{N}$			
C	E	H	J	K	L	M	N	P	R	S

4 a

33	56	93	51	77	91	49	9	89	37	57	19	29	77
56	93	51	77	91	49	33	89	37	57	19	29	77	9
93	56	77	91	51	49	89	37	57	33	29	77	19	9
93	77	91	56	51	89	49	57	37	33	77	29	19	9
93	91	77	56	89	51	57	49	37	77	33	29	19	9
93	91	77	89	56	57	51	49	77	37	33	29	19	9
93	91	89	77	57	56	51	77	49	37	33	29	19	9
93	91	89	77	57	56	77	51	49	37	33	29	19	9
93	91	89	77	57	77	56	51	49	37	33	29	19	9
93	91	89	77	77	57	56	51	49	37	33	29	19	9

So list is:

Amy	93	Annie	51
Greg	91	Harry	49
Janelle	89	Josh	37
Sophie	77	Alex	33
Dom	77	Sam	29
Lucy	57	Myles	19
Alison	56	Hugo	9

4 b

33	56	93	51	77	91	49	9	89	37	57	19	29	77
33	56	93	51	77	91	49	89	37	57	19	29	77	$\boxed{9}$
56	93	51	77	91	89	57	77	$\boxed{49}$	33	37	19	29	
93	$\boxed{91}$	56	51	77	89	57	77		33	37	29	$\boxed{19}$	
$\boxed{93}$		$\boxed{89}$	56	51	77	57	77		$\boxed{37}$	33	29		
			77	$\boxed{77}$	56	51	57			33	$\boxed{29}$		
			$\boxed{77}$		56	57	$\boxed{51}$			$\boxed{33}$			
					$\boxed{57}$	56							
						$\boxed{56}$							
93	91	89	77	77	57	56	51	49	37	33	29	19	9

5 a For n items the maximum number of comparisons made is given by

$$(n-1)+(n-2)+\ldots 3+2+1=\frac{n(n-1)}{2}$$

This occurs when the initial list is in descending order.

b The bubble sort would be quicker, for example, if the items are to be put in increasing order and if the only item out of place is the largest.

c i Bubble sort. Only the 7 is out of place and it will be moved to its final position in the first pass. A second pass is still needed to complete the bubble sort. A total of 11 comparisons is needed for the bubble sort and 14 are needed for the quick sort.

ii Quick sort. This is the worst case for the bubble sort. The 1 is at the wrong end of the list and only moves one place with each pass.

6 a There are 9 names in the list, so the middle will be $\frac{9+1}{2}=5$

The pivot is the 5th name (Mellor) on the list and taken as the pivot. Starting at the beginning of the list, each name is compared with Mellor and placed on the left side if it comes before M or the right side if it comes after M to produce two sub-lists. The process is repeated for each sub-list with pivot of G on the left and M on the right. Select further pivots from within each sub-list and repeat the process.

b

H S F G Me C Mi W A (M is the pivot)
H F G C A $\boxed{\text{Me}}$ S Mi W (1st pass)
F C A $\boxed{\text{G}}$ H $\boxed{\text{Me}}$ $\boxed{\text{Mi}}$ S W (2nd pass)

1st pass: pivots are G and Mi
2nd pass: pivots are C, H and W

Challenge

a Answers will vary.

b When the cards are in order, we have A♥, 2♥... K♥, A♣ ... K♣, A♦ ... K♦, A♠ ... K♠.
Recall that bubble sort compares two cards at a time, swaps them if they are not in order and leaves the pair without change if they are in the required order. We are asked to make a single change to the arrangement above so that bubble sort requires 51 passes to put the cards back in order. In other words, we want the algorithm to go through the whole deck 51 times, each time making at least one swap, before the cards are back in order. This can be achieved by putting the A♥ at the very bottom of the deck. Then at each pass we make exactly one swap, interchanging the positions of A♥ and the card directly before it. The rest of the deck remains unchanged.
For example:
Pass 1: 2♥... K♥, A♣ ... K♣, A♦ ... K♦, A♠ ... <u>K♠, A♥</u>
Changes into: 2♥... K♥, A♣ ... K♣, A♦ ... K♦, A♠ ... <u>A♥, K♠</u>
Pass 2: 2♥... K♥, A♣ ... K♣, A♦ ... K♦, A♠ ... <u>Q♠, A♥,</u> K♠
Changes into: 2♥... K♥, A♣ ... K♣, A♦ ... K♦, A♠ ... <u>A♥, Q♠,</u> K♠
Since we need to move the ace ahead of 51 cards, the algorithm will take exactly 51 passes to put the cards back in order.

Exercise 1E

1 a

Lower bound

$$= \frac{18+4+23+8+27+19+3+26+30+35+32}{50}$$

$$= \frac{225}{50}$$

$$= 4.5$$

Therefore 5 bins (4 bins will be insufficient)

b i Bin 1: 18 + 4 + 23 + 3
Bin 2: 8 + 27
Bin 3: 19 + 26
Bin 4: 30
Bin 5: 35
Bin 6: 32

ii Putting list into descending order

35 32 30 27 26 23 19 18 8 4 3

Bin 1: 35 + 8 + 4 + 3
Bin 2: 32 + 18
Bin 3: 30 + 19
Bin 4: 27 + 23
Bin 5: 26

iii For example
Bin 1: 32 + 18
Bin 2: 27 + 23
Bin 3: 35 + 8 + 4 + 3
Full bins
Bin 4: 19 + 26
Bin 5: 30

2 a Bin 1: A(30) + B(30) + C(30) + D(45) + E(45)
Bin 2: F(60) + G(60) + H(60)
Bin 3: I(60) + J(75)
Bin 4: K(90)
Bin 5: L(120)
Bin 6: M(120)

b Bin 1: M(120) + I(60)
Bin 2: L(120) + H(60)
Bin 3: K(90) + J(75)
Bin 4: G(60) + F(60) + E(45)
Bin 5: D(45) + C(30) + B(30) + A(30)

2 c

Lower bound =

$$\frac{30+30+30+45+45+60+60+60+60+75+90+120+120}{180}$$

$$= \frac{825}{180}$$

= 4.5 so 5 tapes needed at least.

Since a minimum of 5 tapes are needed and **b** uses 5 tapes it is optimal.

d For example
Bin 1: M(120)
Bin 2: L(120)
Bin 3: K(90) + A(30)
Bin 4: G(60) + F(60)
Bin 5: H(60) + I(60)
Bin 6: J(75) + E(45)
Full bins
Bin 7: B(30) + C(30) + D(45)

3 a First-fit does not rely on observation, it takes the items in the order given.
Whereas full-bin uses observation to find combinations of items.

b Bin 1: A(4) + B(7) + C(13) + D(6)
Bin 2: E(13) + F(4) + G(12)
Bin 3: H(14) + I(6)
Bin 4: J(11)
This uses 4 lanes.

c By inspection,
A(4) + B(7) + C(13) + D(6) = 30
E(13) + I(6) + J(11) = 30
F(4) + G(12) + H(14) = 30
Bin 1: A, B, C, D
Bin 2: E, I, J
Bin 3: F, G, H
Each of the three lanes is full, so solution is optimal.

4 a

$$\frac{3+3+4+4+4+4+5+5+5+7+8+8}{15} = 4$$

rolls

b For example,
Bin 1: L(8) J(7)
Bin 2: K(8) I(5)
Bin 3: H(5) G(5) F(4)
Bin 4: E(4) D(4) C(4) B(3)
Bin 5: A(3)
5 rolls used and 15 m wasted.

4 c Doesn't always give an optimal solution.

d For example,
Bin 1: A(3) + C(4) + L(8)
Bin 2: B(3) + D(4) + E(4) + F(4)
Bin 3: G(5) + H(5) + I(5)
Bin 4: J(7) + K(8)
4 rolls used and no carpet is wasted, so solution is optimal.

5 **a** Bin 1: H(25) + A(8)
Bin 2: G(25)
Bin 3: F(24) + B(16)
Bin 4: E(22) + C(17)
Bin 5: D(21)

b

$$\text{Lower bound} = \frac{8+16+17+21+22+24+25+25}{40}$$
$$= \frac{158}{40}$$
$$= 3.95$$

∴ Lower bound is 4.

c There are 5 programs over 20MB. It is not possible for any two of these to share a bin. So at least 5 bins will be needed, so 4 will be insufficient.

Exercise 1F

1 **a** The middle name is the $\left(\frac{8+1}{2}=4.5\right)$ 5^{th} name:

5 Tapner
Connock is before Tapner so the list reduces to:
1 Berry
2 Connock
3 Li
4 Sully

The middle name is the $\left(\frac{4+1}{2}=2.5\right)$ 3rd name:

3 Li
Connock is before Li so the list reduces to:
1 Berry
2 Connock

The middle name in this sublist is the $\left(\frac{2+1}{2}=1.5\right)$ 2nd name:

2 Connock
The search is complete as Connock has been found.

b The middle name is the $\left(\frac{8+1}{2}=4.5\right)$ 5^{th} name:

5 Tapner
Walkey is after Tapner so the list reduces to:
1 Walkey
2 Wilson
3 Wu

The middle name is the $\left(\frac{3+1}{2}=2\right)$ 2nd name:

2 Wilson
Walkey is before Wilson so the list reduces to:
1 Walkey
The search is complete as Walkey has been found.

c The middle name is the $\left(\frac{8+1}{2}=4.5\right)$ 5^{th} name:

5 Tapner
Peabody is before Tapner so the list reduces to:
1 Berry
2 Connock
3 Li
4 Sully

The middle name is the $\left(\frac{4+1}{2}=2.5\right)$ 3rd name:

3 Li
Peabody is after Li, so the list reduces to:
1 Sully
There is only one item on the list and it is not Peabody therefore Peabody is not on the list.

2 a The middle number is the $\left(\frac{12+1}{2}=6.5\right)$ 7^{th} number:

7 15

21 is after 15 so the list reduces to:

1 17

2 18

3 20

4 21

5 24

The middle number is the $\left(\frac{5+1}{2}=3\right)$ 3rd number:

3 20

20 is before 21 so the list reduces to:

1 21

2 24

The middle number is the $\left(\frac{2+1}{2}=1.5\right)$ 2nd number:

2 24

24 is after 21 so the list reduces to:

1 21

The search is complete as 21 has been found.

b The middle number is the $\left(\frac{12+1}{2}=6.5\right)$ 7^{th} number:

7 15

5 is before 15 so the list reduces to:

1 3

2 4

3 7

4 9

5 10

6 13

The middle number is the $\left(\frac{6+1}{2}=3.5\right)$ 4th number:

4 9

5 is before 9 so the list reduces to:

1 3

2 4

3 7

The middle number is the $\left(\frac{3+1}{2}=2\right)$ 2nd number:

2 4

5 is after 4 so the list reduces to:

1 7

The search is complete as 5 has not been found.

3 a Each search reduces the list to half its original size.

So starting with 100 items:

100 50 25 12.5 …

This is a geometric sequence of the form $n = 100(0.5)^s$

Where n is the number of items remaining and s is the number of searches completed.

To find the number of searches required for $n = 1$

$$100(0.5)^s = 1$$
$$0.5^s = 0.01$$
$$s \ln 0.5 = \ln 0.01$$
$$s = \frac{\ln 0.01}{\ln 0.5}$$
$$= 6.643...$$

Therefore 7 searches are required.

3 b Each search reduces the list to half its original size.

So starting with 1000 items:

1000 500 250 125 …

This is a geometric sequence of the form $n = 1000(0.5)^s$

Where n is the number of items remaining and s is the number of searches completed.

To find the number of searches required for $n = 1$

$$1000(0.5)^s = 1$$
$$0.5^s = 0.001$$
$$s \ln 0.5 = \ln 0.001$$
$$s = \frac{\ln 0.001}{\ln 0.5}$$
$$= 9.965...$$

Therefore 10 searches are required.

c Each search reduces the list to half its original size.

So starting with 10 000 items:

10 000 5000 2500 1250 …

This is a geometric sequence of the form $n = 10\,000(0.5)^s$

Where n is the number of items remaining and s is the number of searches completed.

To find the number of searches required for $n = 1$

$$10\,000(0.5)^s = 1$$
$$0.5^s = 0.0001$$
$$s \ln 0.5 = \ln 0.0001$$
$$s = \frac{\ln 0.0001}{\ln 0.5}$$
$$= 13.287...$$

Therefore 14 searches are required.

4 a The correct order is:
Adam, Alex, Des, Doug, Ed, Emily, Felix, George, Hongmei, Jess,
Katie, Lei, Leo, Lottie, Lotus, Matt, Miranda, Oli, Ramin, Saul

b i The middle name is the $\left(\frac{20+1}{2}=10.5\right)$ 11^{th} name:

11 Katie

George is before Katie so the list reduces to:

1 Adam

2 Alex

3 Des

4 Doug

5 Ed

6 Emily

7 Felix

8 George

9 Hongmei

10 Jess

The middle name is the $\left(\frac{10+1}{2}=5.5\right)$ 6^{th} name:

6 Emily

George is after Emily so the list reduces to:

1 Felix

2 George

3 Hongmei

4 Jess

The middle name is the $\left(\frac{4+1}{2}=2.5\right)$ 3rd name:

3 Hongmei

George is before Hongmei so the list reduces to:

1 Felix

2 George

The middle name is the $\left(\frac{2+1}{2}=1.5\right)$ 2nd name:

George

George has been found.

4 b ii The middle name is the $\left(\frac{20+1}{2}=10.5\right)$ 11^{th} name:

11 Katie

David is before Katie so the list reduces to:

1 Adam

2 Alex

3 Des

4 Doug

5 Ed

6 Emily

7 Felix

8 George

9 Hongmei

10 Jess

The middle name is the $\left(\frac{10+1}{2}=5.5\right)$ 6^{th} name:

6 Emily

David is before Emily so the list reduces to:

1 Adam

2 Alex

3 Des

4 Doug

5 Ed

The middle name is the $\left(\frac{5+1}{2}=3\right)$ 3rd name:

3 Des

David is before Des so the list reduces to:

1 Adam

2 Alex

The middle name is the $\left(\frac{2+1}{2}=1.5\right)$ 2nd name:

Alex

David is after Alex.

Since there are no names after Alex, David is not on the list. The search is over.

4 b iii The middle name is the $\left(\frac{20+1}{2}=10.5\right)$ 11^{th} name:

11 Katie

Jess is before Katie so the list reduces to:

1 Adam

2 Alex

3 Des

4 Doug

5 Ed

6 Emily

7 Felix

8 George

9 Hongmei

10 Jess

The middle name is the $\left(\frac{10+1}{2}=5.5\right)$ 6^{th} name:

6 Emily

Jess is after Emily so the list reduces to:

1 Felix

2 George

3 Hongmei

4 Jess

The middle name is the $\left(\frac{4+1}{2}=2.5\right)$ 3rd name:

3 Hongmei

Jess is after Hongmei so the list reduces to:

1 Jess

Jess has been found. The search is over.

Chapter Review 1

1 Bubbling left to right

Initial list	27	15	2	38	16	1
1st pass	15	2	27	16	1	38
2nd pass	2	15	16	1	27	38
3rd pass	2	15	1	16	27	38
4th pass	2	1	15	16	27	38
5th pass	1	2	15	16	27	38

No further changes ∴ sorted

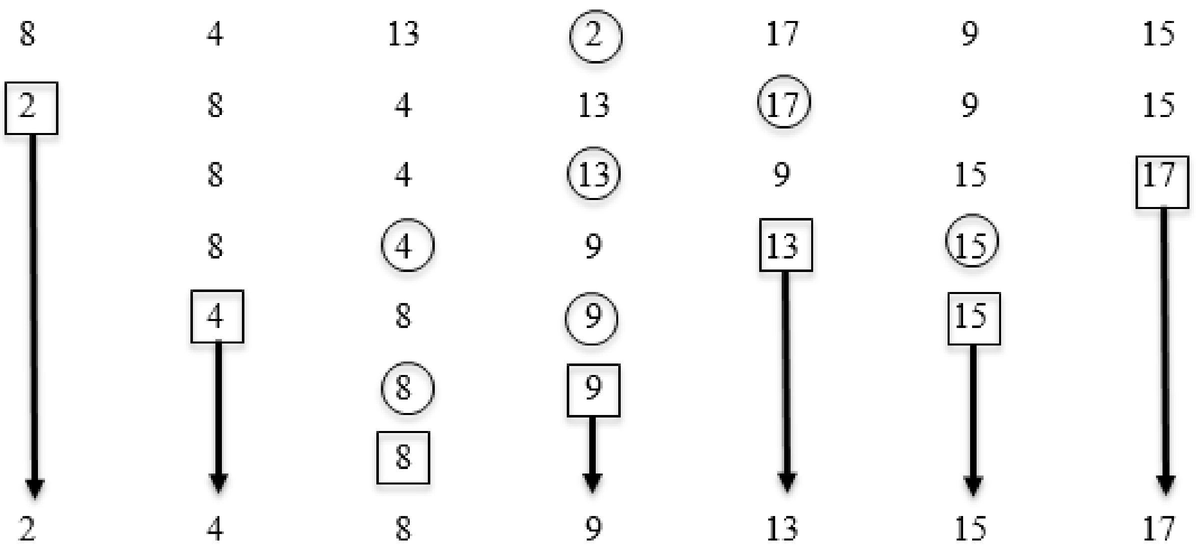
2 a Bubbling left to right

Initial list	25	42	31	22	26	41
1st pass	42	31	25	26	41	22
2nd pass	42	31	26	41	25	22
3rd pass	42	31	41	26	25	22
4th pass	42	41	31	26	25	22

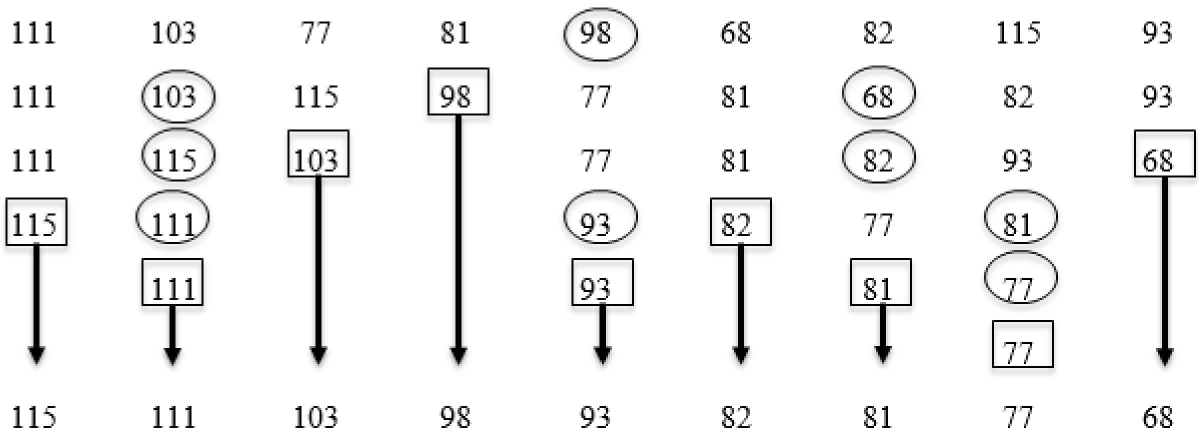
No further changes ∴ sorted

b 15

3



4 a



b i Bin 1: 115 + 82
Bin 2: 111 + 81
Bin 3: 103 + 93
Bin 4: 98 + 77
Bin 5: 68

ii No room in bin 1 (3 left) or bin 2 (8 left) or bin 3 (4 left) but room in bin 4.

5 a Rank the times in descending order and use them in this order. Number the bins starting at 1. Place each recording time into the first available bin, starting with bin 1 each time.

b

100 92 84 75 60 52 42 30

Bin 1: 100
Bin 2: 92
Bin 3: 84 + 30
Bin 4: 75 + 42
Bin 5: 60 + 52

Unused DVDs = 5 × 120 – (100 + 92 + 84 + 75 + 60 + 52 + 42 + 30)
= 600 – 535
= 65 minutes

c There is room on tape 2 for 28 minutes; one of the 25-minute programmes can be recorded on tape 2. But there is no room on any tape for the second programme.

6 a For example, the length total is 12m so no wastage is permitted. We are therefore seeking a full bin solution.
The two 1.2m lengths cannot be 'made up' to 2m bins since these are only $2 \times 0.4\,\mathrm{m}$ length. Two of these can be used to make a full bin, $1.2 + 0.4 + 0.4$, but the second 1.2m cannot be made up to 2m since there is only 1 remaining 0.4 m length.

b Bin 1: 1.6 + 0.6
Bin 2: 1.4 + 1
Bin 3: 1.2 + 1.2
Bin 4: 1 + 1 + 0.4
Bin 5: 0.6 + 0.6 + 0.6 + 0.4
Bin 6: 0.4

6 c For example:
Bin 1: 1.6 + 0.4 + 0.4
Bin 2: 1.4 + 1
Bin 3: 1.2 + 1.2
Bin 4: 1 + 1 + 0.4
Bin 5: 0.6 + 0.6 + 0.6 + 0.6

7 a

	I	***M***	**Box 4**	***A***	**Temp**	**Box 6**
Initial conditions	1	-		6.1	1.1	
1st pass	2	1.9	No	6.1	1.1	Yes
2nd pass	3	0.7	Yes	5.7	0.7	Yes
3rd pass	4	0.2	Yes	4.8	0.2	Yes
4th pass	5	0.3	No			No

Output = 4.8

b It selects the number nearest to 5.

c It would select the number furthest from 5.

8 a Bubbling left to right
After 1st pass: 2.0 1.3 1.6 0.3 1.3 0.3 0.2 2.0 0.5 0.1
After 2nd pass: 2.0 1.6 1.3 1.3 0.3 0.3 2.0 0.5 0.2 0.1
After 3rd pass: 2.0 1.6 1.3 1.3 0.3 2.0 0.5 0.3 0.2 0.1
After 4th pass: 2.0 1.6 1.3 1.3 2.0 0.5 0.3 0.3 0.2 0.1
After 5th pass: 2.0 1.6 1.3 2.0 1.3 0.5 0.3 0.3 0.2 0.1
After 6th pass: 2.0 1.6 2.0 1.3 1.3 0.5 0.3 0.3 0.2 0.1
After 7th pass: 2.0 2.0 1.6 1.3 1.3 0.5 0.3 0.3 0.2 0.1
After 8th pass: 2.0 2.0 1.6 1.3 1.3 0.5 0.3 0.3 0.2 0.1
No swap in 8th pass, so the list is in descending order.

b Sorting into descending order, 2.0, 2.0, 1.6, 1.3, 1.3, 0.5, 0.3, 0.3, 0.2, 0.1
Bin 1: 2.0
Bin 2: 2.0
Bin 3: 1.6 + 0.3 + 0.1
Bin 4: 1.3 + 0.5 + 0.2
Bin 5: 1.3 + 0.3
5 lengths of pipe needed

c Yes: The lower bound is given by $\frac{2.0+2.0+1.6+1.3+1.3+0.5+0.3+0.3+0.2+0.1}{2} = \frac{9.6}{2} = 4.8$
rounded up to 5 lengths of pipe.

9

M	V	C	A	D	B	K	S
C	A	B	$\boxed{D}$	M	V	K	S
$\boxed{A}$	C	B	$\boxed{D}$	$\boxed{K}$	M	V	S
$\boxed{A}$	$\boxed{B}$	C	$\boxed{D}$	$\boxed{K}$	M	S	$\boxed{V}$
$\boxed{A}$	$\boxed{B}$	$\boxed{C}$	$\boxed{D}$	$\boxed{K}$	M	$\boxed{S}$	$\boxed{V}$

There are 8 names in the list, so the pivot should be the name to the right of the middle (Daisy). Starting at the beginning of the list, each name is compared with Daisy and placed on the left side if it comes before D or the right side if it comes after D to produce two sub lists. The process is repeated for each sub-list with pivot of A on the left and K on the right. Select further pivots from within each sub-list and repeat the process until the names are in alphabetical order.

Challenge

a The names are not in ascending alphabetical order and so a binary search cannot be done.

b Starting at the beginning of the list, compare the first two letters. If they are in alphabetical order, leave them in position, otherwise swap them. Continue through the list, to the end, comparing every pair of letters in the same way.
J M C B T H K R G F
J C B M H K R G F T
C B J H K M G F R T
B C H J K G F M R T
B C H J G F K M R T
B C H G F J K M R T
B C G F H J K M R T
B C F G H J K M R T
Bubble sort.
(Alternatively, quick sort could have been used to achieve the same result.)

c The middle name is the $\left(\frac{10+1}{2}=5.5\right)$ 6^{th} name:

6 Jenny
Kim is after Jenny so the list reduces to:
1 Kim
2 Merry
3 Richard
4 Toby

The middle name is the $\left(\frac{4+1}{2}=2.5\right)$ 3rd name:

3 Richard
Kim is before Richard so the list reduces to:
1 Kim
2 Merry

The middle name in this sublist is the $\left(\frac{2+1}{2}=1.5\right)$ 2nd name:

2 Merry
The list reduces to
1 Kim
The search is complete as Kim has been found.